Quadratics (1)

1. **(a)** State the relation between a, b and c such that the equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has equal roots.

(b) If the equation $a^2x^2 + 3abx + ac + 2b^2 = 0$ $(a \neq 0)$ has equal roots, show that the roots for the equation $ac(x+1)^2 = b^2x$ $(a, b, c \neq 0)$ are equal.

(a) The condition for the equation $ax^2 + bx + c = 0$ has equal roots is $\Delta = b^2 - 4ac = 0$

(b) If the equation $a^2x^2 + 3abx + 2b^2 = 0$ has equal roots,

$$\Delta = (3ab)^2 - 4a^2(ac + 2b^2) = 0 \quad (a \neq 0)$$
$$\therefore b^2 = 4ac$$

Method 1

For the equation $ac(x+1)^2 = b^2x \Leftrightarrow ac(x^2 + 2x + 1) - b^2x = 0$

$$\Leftrightarrow acx^2 + (2ac - b^2)x + ac = 0$$

Since $\Delta = (2ac - b^2)^2 - 4(ac)(ac) = b^4 - 4acb^2 = b^2(b^2 - 4ac) = 0$

Therefore $ac(x+1)^2 = b^2x$ has equal roots.

Method 2

Since $a, b, c \neq 0$

$$ac(x+1)^2 = b^2x \Leftrightarrow ac(x+1)^2 = 4acx \Leftrightarrow (x+1)^2 = 4x \Leftrightarrow x^2 + 2x + 1 = 4x \Leftrightarrow (x-1)^2 = 0$$

$\therefore$ $x = 1$ (equal roots)

2. If x is real, show that the expression $y = \frac{x^2+x+1}{x+1}$ does not have a value between -3 and 1.

$$y = \frac{x^2+x+1}{x+1} \Leftrightarrow y(x+1) = x^2 + x + 1 \Leftrightarrow x^2 + (1-y)x + (1-y) = 0$$

Since x is real, $\Delta = (1-y)^2 - 4(1-y) \geq 0$

$$y^2 + 2y - 3 \geq 0$$
$$(y-1)(y+3) \geq 0$$
$$\begin{cases} y - 1 \leq 0 \\ y + 3 \leq 0 \end{cases} \text{ or } \begin{cases} y - 1 \geq 0 \\ y + 3 \geq 0 \end{cases}$$
$$\therefore y \leq -3 \text{ or } y \geq 1$$

3. Let the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ $(b \neq d)$ have one non-zero common root. Form an equation with the other roots of these equations.

Let α be the non-zero common root. Then by Vieta Theorem the other roots are $\frac{b}{\alpha}$ and $\frac{d}{\alpha}$.

Also $\alpha^2 + a\alpha + b = 0$ and $\alpha^2 + c\alpha + d = 0$ and their difference is

$$(a-c)\alpha + (b-d) = 0 \text{ giving } \alpha = \frac{d-b}{a-c} \quad (\text{check that } a \neq c)$$

Thus, the other roots are $\frac{b(a-c)}{d-b}$ and $\frac{d(a-c)}{d-b}$.

Sum of roots = $\frac{b(a-c)}{d-b} + \frac{d(a-c)}{d-b} = \frac{(b+d)(a-c)}{d-b}$

Product of roots = $\frac{b(a-c)}{d-b} \times \frac{d(a-c)}{d-b} = \frac{bd(a-c)}{(d-b)^2}$

Therefore the required equation is $x^2 - \frac{(b+d)(a-c)}{d-b}x + \frac{bd(a-c)}{(d-b)^2} = 0$

4. If a, b and c are real numbers, show that the roots of the equation $(a-b-c)x^2+ax+b+c=0$ is real. If one of the roots is twice the other, show that $b+c=\frac{a}{3}$ or $\frac{2a}{3}$.

For the equation $(a-b-c)x^2+ax+b+c=0$

$$\Delta = a^2-4(a-b-c)(b+c) = a^2-4[a-(b+c)](b+c) = a^2-4a(b+c)+4(b+c)^2$$
$$= [a-2(b+c)]^2 \geq 0$$

Hence, the roots of the equation $(a-b-c)x^2+ax+b+c=0$ is real.

If one of the roots is twice the other, let $\alpha, 2\alpha$ be the roots.

By Vieta Theorem, $\alpha+2\alpha = -\frac{a}{a-b-c}$ ... (1)

$(\alpha)(2\alpha) = \frac{b+c}{a-b-c}$ ...(2)

From (1), $\alpha = -\frac{a}{3[a-(b+c)]}$ ... (3)

From (2), $2\alpha^2 = \frac{b+c}{a-(b+c)}$ ... (4)

(3) ↓ (4), $2\left\{-\frac{a}{3[a-(b+c)]}\right\}^2 = \frac{b+c}{a-(b+c)}$

$$\frac{2a^2}{9[a-(b+c)]} = b+c$$
$$2a^2 = 9a(b+c)-9(b+c)^2$$
$$9(b+c)^2-9a(b+c)+2a^2 = 0$$
$$[3(b+c)-a][3(b+c)-2a] = 0$$
$$\therefore\ b+c = \frac{a}{3} \text{ or } \frac{2a}{3}$$

5. If α_1 and β_1 are the roots of the equation $x^2+2ax+b^2=0$ and α_2 and β_2 are the roots of the equation $x^2+2cx+d^2=0$, show that :

(a) If $\alpha_1+\alpha_2 = \beta_1+\beta_2$, then $a^2+d^2=b^2+c^2$,
(b) If $\alpha_1\alpha_2+\beta_1\beta_2=0$, then $b^2d^2=a^2d^2+b^2c^2$.

(a) By Vieta Theorem, $\alpha_1+\beta_1=-2a,\ \alpha_1\beta_1=b^2$ and $\alpha_2+\beta_2=-2c,\ \alpha_2\beta_2=d^2$

Since $\alpha_1+\alpha_2 = \beta_1+\beta_2$

$$\alpha_1-\beta_1 = -(\alpha_2-\beta_2)$$
$$(\alpha_1-\beta_1)^2 = (\alpha_2-\beta_2)^2$$
$$\alpha_1{}^2-2\,\alpha_1\,\beta_1+\beta_1{}^2 = \alpha_2{}^2-2\,\alpha_2\,\beta_2+\beta_2{}^2$$
$$(\alpha_1+\beta_1)^2-4\,\alpha_1\,\beta_1 = (\alpha_2+\beta_2)^2-4\alpha_2\,\beta_2$$
$$(-2a)^2-4\,b^2 = (-2c)^2-4d^2$$
$$a^2+d^2 = b^2+c^2$$

(b) $a^2d^2+b^2c^2 = \left(-\frac{\alpha_1+\beta_1}{2}\right)^2(\alpha_2\beta_2)+(\alpha_1\beta_1)\left(-\frac{\alpha_2+\beta_2}{2}\right)^2 = \frac{\alpha_1{}^2+2\,\alpha_1\,\beta_1+\beta_1{}^2}{4}(\alpha_2\beta_2)+(\alpha_1\beta_1)\frac{\alpha_2{}^2+2\,\alpha_2\,\beta_2+\beta_2{}^2}{4}$

$$= \frac{1}{4}\left(4\,\alpha_1\alpha_2\,\beta_1\beta_2+\alpha_1\beta_1\,\alpha_2{}^2+\alpha_1\,\beta_1\,\beta_2{}^2+\alpha_2\beta_2\,\alpha_1{}^2+\alpha_2\,\beta_2\,\beta_1{}^2\right)$$
$$= \frac{1}{4}[4\,\alpha_1\alpha_2\,\beta_1\beta_2+(\alpha_1\alpha_2+\beta_1\beta_2)\alpha_1\beta_2+(\alpha_1\alpha_2+\beta_1\beta_2)\alpha_2\beta_1] = \frac{1}{4}[4\,\alpha_1\alpha_2\,\beta_1\beta_2+(0)\alpha_1\beta_2+(0)\alpha_2\beta_1]$$
$$= \alpha_1\alpha_2\,\beta_1\beta_2 = (\alpha_1\beta_1)(\alpha_2\beta_2) = b^2d^2$$

6. If the equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has real roots, show that the equation

$$(a + c - b)x^2 - 2(a - c)x + (a + c + b) = 0$$

has also real roots.

Show that if α and β are the roots for the first equation, then the product of roots of the second equation is

$$\frac{(1-\alpha)(1-\beta)}{(1+\alpha)(1+\beta)}$$

The equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has real roots if and only if $b^2 - ac \geq 0$

For the equation $(a + c - b)x^2 - 2(a - c)x + (a + c + b) = 0$,

$$\begin{aligned} \Delta &= [-2(a-c)]^2 - 4(a+c-b)(a+c+b) \\ &= 4(a-c)^2 - 4[(a+c)^2 - b^2] \\ &= 4[(a-c)^2 - (a+c)^2] + 4b^2 \\ &= 4(-4ac) + 4b^2 \\ &= 4(b^2 - 4ac) \geq 0 \end{aligned}$$

Therefore the second equation has real roots.

For the first equation, by Vieta Theorem, $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

The product of roots of the second equation is

$$\frac{a+c+b}{a+c-b} = \frac{1+\frac{c}{a}+\frac{b}{a}}{1+\frac{c}{a}-\frac{b}{a}} = \frac{1+\alpha\beta-(\alpha+\beta)}{1+\alpha\beta+(\alpha+\beta)} = \frac{(1-\alpha)(1-\beta)}{(1+\alpha)(1+\beta)}$$

7. If α, β are roots of the equation $x^2 + px + q = 0$ and α_1, β_1 are roots of the equation $x^2 + p_1x + q_1 = 0$.
Express $(\alpha - \alpha_1)(\alpha - \beta_1) + (\beta - \alpha_1)(\beta - \beta_1)$ in terms of p, q, p_1 and q_1.

By Vieta Theorem, $\alpha + \beta = -p$, $\alpha\beta = q$

$\alpha_1 + \beta_1 = -p_1$, $\alpha_1\beta_1 = q_1$

$$\begin{aligned} &(\alpha - \alpha_1)(\alpha - \beta_1) + (\beta - \alpha_1)(\beta - \beta_1) \\ &= \alpha^2 + \beta^2 - \alpha(\alpha_1 + \beta_1) - \beta(\alpha_1 + \beta_1) + 2\alpha_1\beta_1 \\ &= (\alpha + \beta)^2 - 2\alpha\beta - (\alpha + \beta)(\alpha_1 + \beta_1) + 2\alpha_1\beta_1 \\ &= (-p)^2 - 2q - (-p)(-p_1) + 2q_1 \\ &= p^2 - p\,p_1 - 2q + 2q_1 \end{aligned}$$

8. Show that the expression $\frac{5}{2x^2+3x+3}$ is positive and find its greatest value.
Hence find the smallest values of $\frac{6x^2+9x+4}{2x^2+3x+3}$.
Sketch the functions of $2x^2 + 3x + 3$, $\frac{5}{2x^2+3x+3}$, $\frac{6x^2+9x+4}{2x^2+3x+3}$ together on the same graph.

$$2x^2 + 3x + 3 = 2\left[x^2 + \frac{3}{2}x + \frac{3}{2}\right] = 2\left[x^2 + 2\left(\frac{3}{4}\right)x + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 + \frac{3}{2}\right] = 2\left[\left(x + \frac{3}{2}\right)^2 - \frac{9}{16} + \frac{3}{2}\right]$$

$$= 2\left[\left(x + \frac{3}{4}\right)^2 + \frac{15}{16}\right] > 0$$

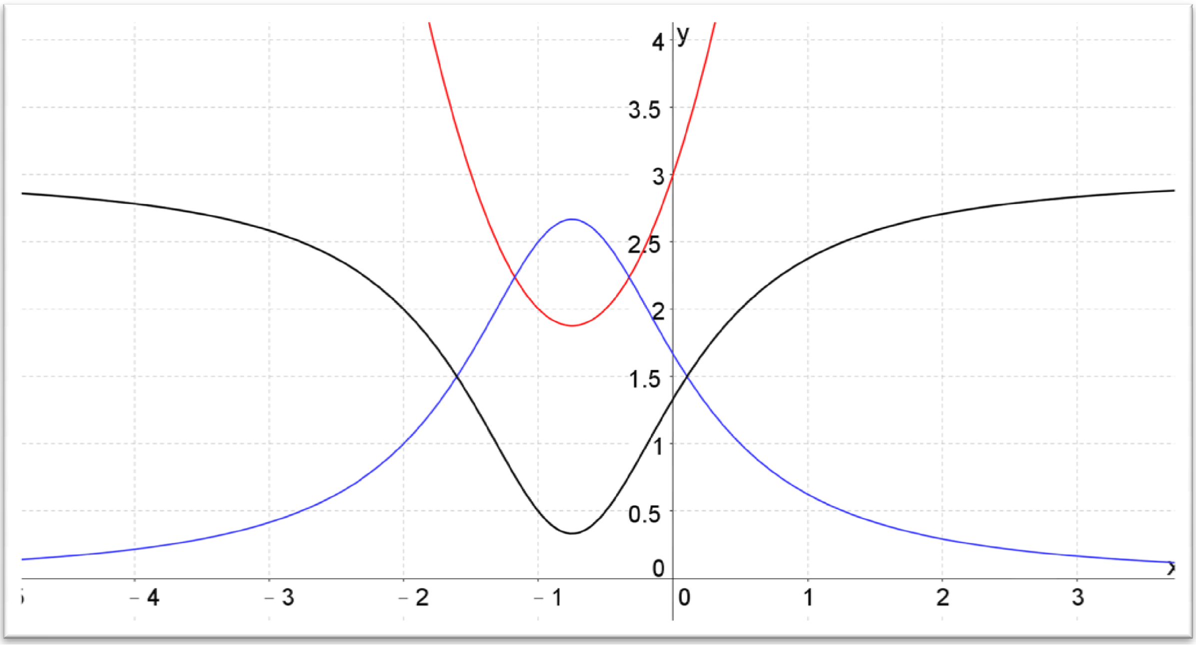
Therefore $\frac{5}{2x^2+3x+3}$ is positive.

$$\text{Min}\{2x^2+3x+3\} = \text{Min}\left\{2\left[\left(x+\frac{3}{4}\right)^2+\frac{15}{16}\right]\right\} = 2\left[0+\frac{15}{16}\right] = \frac{15}{8}$$

$$\therefore \text{Max}\left\{\frac{5}{2x^2+3x+3}\right\} = \frac{5}{\frac{15}{8}} = \frac{8}{3}$$

$$\frac{6x^2+9x+4}{2x^2+3x+3} = \frac{3(2x^2+3x+3)-5}{2x^2+3x+3} = 3-\frac{5}{2x^2+3x+3}$$

$$\therefore \text{Min}\left\{\frac{6x^2+9x+4}{2x^2+3x+3}\right\} = 3-\frac{8}{3} = \frac{1}{3}$$



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